Nýjar aðferðir við túlkun vaxtagagna: New methods for analysis of interest rate data

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- Taylor expansion of Kolmogorov forward equation

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- Description of calculation of approximations

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Background

- Financial data, tick-by-tick

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- Estimation of diffusion models

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- Answer: A stochastic model describing continuous time dynamics. Such models have been made popular in mathematical fincance, e.g., by Nobel-prize winners Merton and Scholes.
- A typical way of representing such models, i.e. desribing the nature of a dynamic process $X(t)$, is by means of stochastic differential equations:

$$dX(t) = \underbrace{\mu(X(t), \boldsymbol{\theta})dt}_{\text{drift}} + \underbrace{\sigma(X(t), \boldsymbol{\theta})dW(t)}_{\text{diffusion term}}$$

$\boldsymbol{\theta}$ is að vector of parameters that define the behaviour of the process.

- $W(t)$ is a standard Wiener process. For $t > s$, $E(W(t)|W(s)) = W(s)$, W continuous independent increment process. $dW(t)$ is continuous time white noise.
- The $\mu(X(t), \boldsymbol{\theta})dt$ part represents the predictable part of the process.
- The $\sigma(X(t), \boldsymbol{\theta})dW$ part represents the stochastic part.

Data and model

- The mathematical idealization

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- A given diffusion:

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- A few approaches, simulation methods, method of moments, estimating functions, maximum-likelihood approximation.

- The transition density:

  $$f(x|x_0, \Delta) = \text{ density for } X(t+\Delta) \text{ given } X(t) = x_0$$

  is only known for for some specific $\mu(X(t), \boldsymbol{\theta})$ and $\sigma(X(t), \boldsymbol{\theta})$

- Therefore maximizing the log-likelihood, i.e., solving:

  $$\max_{\boldsymbol{\theta}} l(\boldsymbol{\theta}|X(t_1), \ldots, X(t_n))$$

  is only possible in some special cases.

Some populuar diffusion models

$$\text{OU} \quad dX(t) = \kappa(\alpha - X(t))dt + \sigma dW(t) \tag{1}$$

Ornstein-Uhlenbeck/Vasicek

$$\text{CIR} \quad dX(t) = \kappa(\alpha - X(t))dt + \sigma\sqrt{X(t)}dW(t) \tag{2}$$

Cox-Ingersoll-Ross/square-root process

$$\text{CKLS} \quad dX(t) = \kappa(\alpha - X(t))dt + \sigma X(t)^\rho dW(t) \tag{3}$$

Chan, Karolyi, Longstaff & Sanders (1992),
Cases of special interest $\rho = 1/2$ and $\rho = 1$

These are all stochastic versions of a very simple differential equation:

$$dX(t) = \kappa(\alpha - X(t))dt$$

Given $X(0)$, the solution is of the form:

$$X(t) = \alpha + \exp(-\kappa t)(X(0) - \alpha)$$

The α parameter is the longtime equilibrium, κ controls the speed of convergence to equilibrium.

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In the stochastic case σ and ρ control the volality of the process.

Kolmogorov forward equation

- What is known about $f(x|x_0, \Delta)$?, $x = x(t + \Delta)$, $x_0 = x(t)$

Kolmogorov forward equation

- What is known about $f(x|x_0, \Delta)$?, $x = x(t + \Delta)$, $x_0 = x(t)$
- Since $X(t)$ is a diffusion process the density function $f(x|x_0, \Delta)$ solves:

$$\frac{\partial f(x|x_0, \Delta)}{\partial \Delta} + \frac{\partial \left(\mu(x, \boldsymbol{\theta}) f(x|x_0, \Delta)\right)}{\partial x}$$
$$-\frac{1}{2}\frac{\partial^2 \left(\sigma^2(x, \boldsymbol{\theta}) f(x|x_0, \Delta)\right)}{\partial x^2} = 0$$

What can be said about it?

Assuming $\sigma(x) = 1$,

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$$\frac{\partial p(x|x_0, \Delta)}{\partial \Delta} + \mu'(x) + \mu(x)\frac{\partial p(x|x_0, \Delta)}{\partial x} \tag{4}$$
$$-\frac{1}{2}\left[\frac{\partial p(x|x_0, \Delta)}{\partial x}\right]^2 - \frac{1}{2}\frac{\partial^2 p(x|x_0, \Delta)}{\partial x^2} = 0$$

A Taylor expansion of $p(x|x_0, \Delta)$ in Δ is on the form:

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$$-\frac{1}{2}log(2\pi\Delta) - \frac{(x-x_0)^2}{2\Delta} + c_0(x|x_0) + c_1(x|x_0)\Delta$$
$$+c_2(x|x_0)\frac{\Delta^2}{2} + c_3(x|x_0)\frac{\Delta^3}{3!} + \cdots$$

Substituting the Taylor expansion into equation (4) gives the first two terms:

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$$- \frac{(x - x_0)(\mu(x) - c_0'(x|x_0))}{\Delta} \tag{5}$$

$$-\frac{1}{2}c_0(x|x_0)'(x)^2 + \mu(x) + \mu(x)c_0'(x|x_0) \tag{6}$$

$$-\frac{c_0''(x|x_0)}{2} + c_1(x|x_0) + (x - x_0)c_1'(x|x_0)$$

Next terms

Next terms

$$+\frac{1}{2}\Delta(2(\mu(x) - c_0'(x|x_0))c_1'(x|x_0) - c_1''(x|x0)$$

$$+2c_2(x|x_0) + (x - x_0)c_2'(x|x0))$$

$$+\frac{1}{12}\Delta^2(-6c_1'(x|x_0)^2 + 6\left(\mu(x) - c_0'(x|x_0)\right)c_2'(x|x_0)$$

$$-3c_2''(x|x_0) + 6c_3(x|x_0) + 2(x - x_0)c_3'(x|x_0))$$

$$\vdots$$

The Kolomogorov equations force the coefficients for each power of Δ to be zero.

Equation (5) gives

$$c_0(x|x_0) = \int_{x_0}^{x} \mu(u)du$$

- Substituting $c_0(x|x_0)$ into reduces the system of equations to

$$\frac{1}{2}(\mu(x)^2 + \mu'(x)^2) + c_1(x|x_0) + (x - x0)c_1'(x|x_0) = 0$$

$$-c_1''(x|x0) + 2c_2(x|x_0) + (x - x_0)c_2'(x|x0) = 0$$

$$-\frac{3}{2}c_2''(x|x_0) - 3c_1'(x|x_0)^2$$

$$+3c_3(x|x_0) + (x - x_0)c_3'(x|x_0) = 0$$

$$\vdots$$

And more

And more

$$-2c_3''(x|x_0) - 12c_1'(x)c_2'(x) +$$

$$4c_4(x|x_0) + (x - x_0)c_4'(x|x_0) = 0$$

$$-\frac{5}{2}c_4''(x|x_0) - 20c_1'(x|x_0)c_3'(x|x_0)$$

$$-15c_2'(x|x_0)^2 + 5c_5(x|x_0) + (x - x_0)c_5'(x|x_0) = 0$$

$$\vdots$$

- The c_j functions are derived recursively by solving the differential equations of the type:

$$jc_j(x|x_0) + (x - x_0)c_j'(x|x0) = q_j(x) \qquad \text{which gives}$$

$$c_j(x) = \frac{1}{(x - x_0)^j} \int_{x_0}^{x} (u - x_0)^{j-1} q_j(u) du$$

- The functions $q_j(x)$ are deciced by $c_0, \ldots, c_{j-1}$

Some comments

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$$Y(t) = \gamma(X(t)) = \pm \int^{X(t)} \frac{du}{\sigma(u)} + \text{constant}$$

- Ito's lemma gives that $Y(t)$ will have unit diffusion and drift:

$$\mu_Y(y) = \pm \left(\frac{\mu(\gamma^{-1}(y))}{\sigma(\gamma^{-1}(y))} - \frac{1}{2} \frac{\partial \sigma}{\partial x}(\gamma^{-1}(y)) \right)$$

- The densities f_X and f_Y are related by:

$$f_X(x|x_0,\Delta) = f_Y(y|y_0,\Delta)|\text{Jacobian}| = f_Y(y|y_0,\Delta)/\sigma(x)$$

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- I.e. the connection between the log-densites for $X(t)|X(0)$ and $Y(t)|Y(0)$ is:

$$p_X(x|x_0, \Delta) = p_Y(y|y_0, \Delta) + log(1/\sigma(x))$$

- Therefore the transformation is only a technicality. I.e. the Taylor coefficints c_j will be messier functions of x and x_0 than of $y = \gamma(x)$ and $y_0 = \gamma(x_0)$.

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- Numerically it might be better to work with $Y(t)$ than $X(t)$.

- Solving recursively for the functions c_j becomes increasingly complicated. A Taylor expansion in x around x_0 (or y around y_0) is therefore an option.

- If no transformation took place, one still gets a sequence of differential equations to solve, but they will be more complicated.

$$2c_{-1}(x) + \sigma(x)^2(c'_{-1}(x))^2 = 0 \quad c_{-1}(x_0) = 0$$

will be the first one

Recursive system for untransformed variable:

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Condition (1) $v(x) = \sigma^2(x)$

$$v(x)c'_{-1}(x)^2 + 2c_{-1}(x) = 0$$

$$c_{-1}(x) = -\frac{1}{2}\left(\int_{x_0}^{x} \frac{du}{\sqrt{v(x)}}\right)^2$$

Condition(2)

$$-c'_0(x)c'_{-1}(x)v(x)^2 - \frac{1}{2}c''_{-1}(x)v(x)^2$$
$$-\frac{3}{2}c'_{-1}(x)v'(x)v(x) + \mu(x)c'_{-1}(x) - \frac{1}{2} = 0$$

Condition (3) (Coefficient on t)

$$c_1(x) - c_1'(x)c_1'(x)\sigma(x)^2 + \mu(x)c_0'(x) + \mu'(x) -$$
$$\frac{\mu(x)\sigma'(x)}{2\sigma(x)}$$
$$-\frac{1}{2}c_0'(x)^2\sigma(x)^2 - \frac{1}{2}c_0''(x)\sigma(x)^2 - \frac{3}{2}c_0'(x)\sigma'(x)\sigma(x) -$$
$$\frac{3}{4}\sigma''(x)\sigma(x) - \frac{3}{8}\sigma'(x)^2 = 0$$

Higher dimensions

- Same principles apply

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- Same principles apply
- Think of a 2-dimensional case.

$$d\begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix} = \begin{bmatrix} \mu_1(X_1(t), X_2(t), \boldsymbol{\theta}) \\ \mu_2(X_1(t), X_2(t), \boldsymbol{\theta}) \end{bmatrix} dt +$$

$$\begin{bmatrix} \sigma_{11}(X_1(t), X_2(t), \boldsymbol{\theta}) & \sigma_{12}(X_1(t), X_2(t), \boldsymbol{\theta}) \\ \sigma_{21}(X_1(t), X_2(t), \boldsymbol{\theta}) & \sigma_{21}(X_1(t), X_2(t), \boldsymbol{\theta}) \end{bmatrix} \begin{bmatrix} dW_1(t) \\ dW_2(t) \end{bmatrix}$$

- The log-density is assumed to be of the form

$$-\frac{m}{2}log(2\pi\Delta) - D(\boldsymbol{x},\boldsymbol{\theta}) + \frac{C^{(-1)}(\boldsymbol{x},\boldsymbol{\theta})}{\Delta} +$$
$$C^{(0)}(\boldsymbol{x},\boldsymbol{\theta}) + C^{(1)}(\boldsymbol{x},\boldsymbol{\theta})\Delta + C^{(2)}(\boldsymbol{x},\boldsymbol{\theta})\Delta^2/2 +$$
$$C^{(3)}(\boldsymbol{x},\boldsymbol{\theta})\Delta^3/3! + \cdots$$
$$D(\boldsymbol{x},\boldsymbol{\theta}) = \frac{1}{2}log(det(\sigma(\boldsymbol{x},\boldsymbol{\theta})\sigma(\boldsymbol{x},\boldsymbol{\theta})^T)$$

- Some conditions on σ_{ij} are needed in order to find a neat transformation as in the univariate case.

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- Ait-Sahalia calls that situation a „reducible” case, and the case where such a transformation does not exist.

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- The resulting functions will be $C^{(j,l)}$ where an l-th order Taylor approximation of order l has been taken in x. A trick that could also be useful in one dimension.

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- The resulting functions will be $C^{(j,l)}$ where an l-th order Taylor approximation of order l has been taken in x. A trick that could also be useful in one dimension.

- The $C^{(j,l)}$ functions can be derived analogously to the univariate case in a messy but straightforeward manner.

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A small simulation. The model

$$dX = \kappa(\alpha - X)dt + \sigma X^{\rho} dW$$

is simulated using Milstein-scheme (strong Taylor of order 1, 25 replications).

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- The time spans used are T= 1, 10, 100
- Δ=1, 0.1 and 0.01 are used.
- 10 points of process per observation.
- $\kappa = 0.24$, $\alpha = 0.07$, $\sigma = 0.08838$, $\rho = 0.75$.

	T=1	T=10	T=100
$\hat{\kappa}$	3.2742	0.5352	0.2709
$\hat{\alpha}$	0.0793	0.0962	0.0695
$\hat{\sigma}$	0.1115	0.0979	0.0899
$\hat{\rho}$	0.7695	0.7732	0.7570

Table 1: Average estimates, for Δ=0.01

	delta=1	delta=0.1
$\hat{\kappa}$	0.8232	0.2916
$\hat{\alpha}$	0.0644	0.0744
$\hat{\sigma}$	0.0984	0.0864
$\hat{\rho}$	0.7342	0.7299

Table 2: Average estimates, for T=100

	T=1	T=10	T=100
s.d. $\hat{\kappa}$	1.8703	0.3995	0.0693
s.d $\hat{\alpha}$	0.0234	0.1269	0.0048
s.d. $\hat{\sigma}$	0.0542	0.0309	0.0042
s.d $\hat{\rho}$	0.2794	0.1084	0.0176

Table 3: Standard deviation of simulations, for Δ=0.01

	delta=1	delta=0.1
s.d. $\hat{\kappa}$	0.4680	0.1807
s.d $\hat{\alpha}$	0.0099	0.0248
s.d. $\hat{\sigma}$	0.0510	0.0196
s.d $\hat{\rho}$	0.2190	0.0851

Table 4: Standard deviation of simulations, for T=100

A two dimensional example:

$$dX = \mu dt + \sigma_1 e^Y dW_1$$

$$dY = \kappa(\alpha - Y)dt + \sigma_2 dW_2$$

Results of 20 replications, $\mu = 0$, $\sigma_1 = \sigma_2 = 1$, $\kappa = 20$, $\alpha = 0.01$. T=1, $\Delta = 1/10000$

	$\hat{\kappa}$	$\hat{\alpha}$	$\hat{\sigma}_1$	$\hat{\mu}$	$\hat{\sigma}_2$
$\bar{m}$	24.1	0.0053	1.004	−0.159	1.000
sd	5.61	0.0519	0.006	0.983	0.007

Table 5: Simulation of 2-dim. model

Icelandic interest rate data

- Each transaction in 2002-2004.
- Zero-coupon governmental bonds, annualized to $r(t)$.
- Form of data:

RIKV	02	06	05	96.165	03.01.2002	11:42:4
RIKV	02	08	06	94.915	04.01.2002	11:13:1

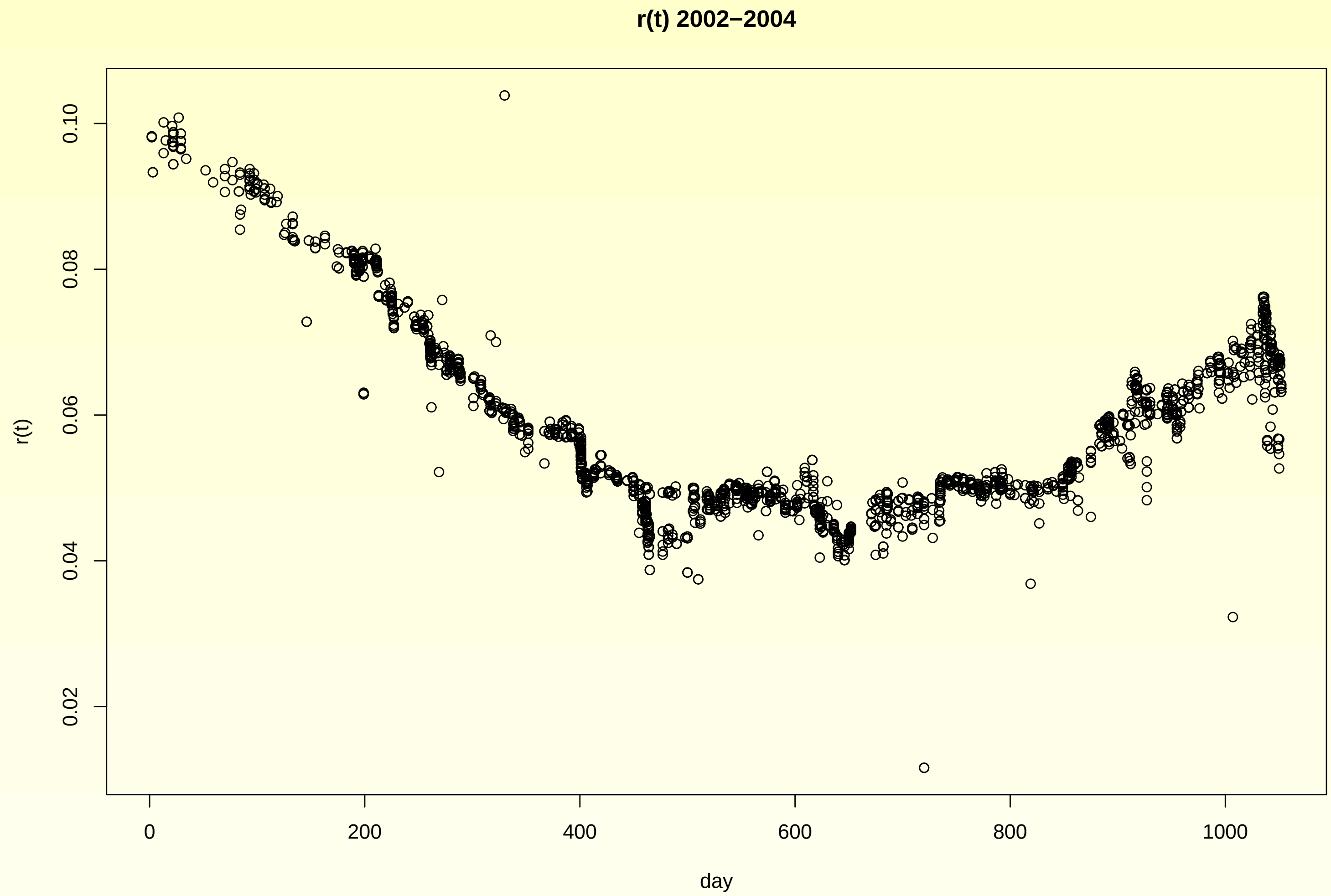

Figure 1: 1000 days of Icelandic bond market

- Timeperiod is 1050 days. Trading took place on 432 days.

- 1933 transactions took place 5 days a week.

1	2	3	4	5
383	410	351	452	337

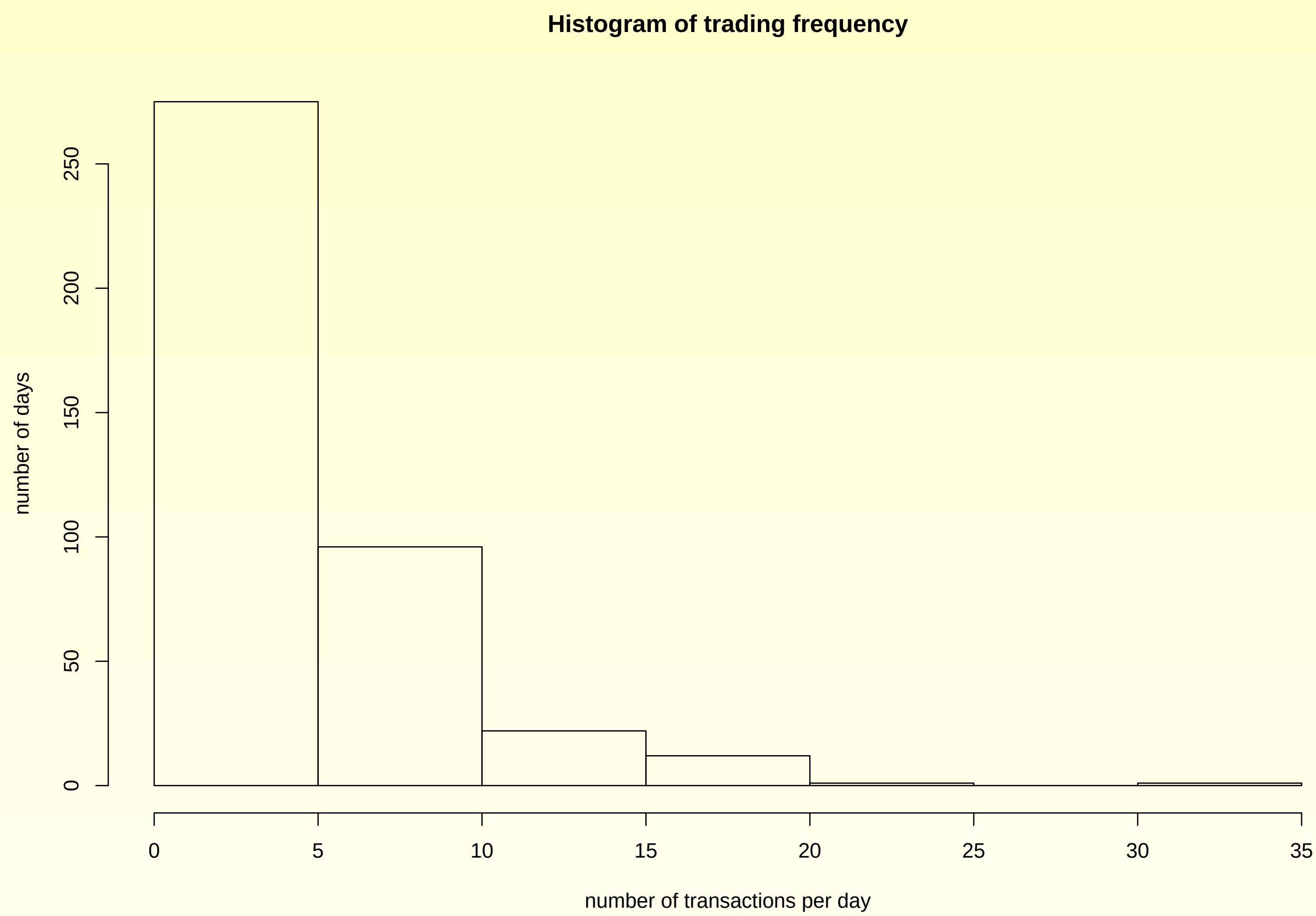

Figure 2: Trading frequency

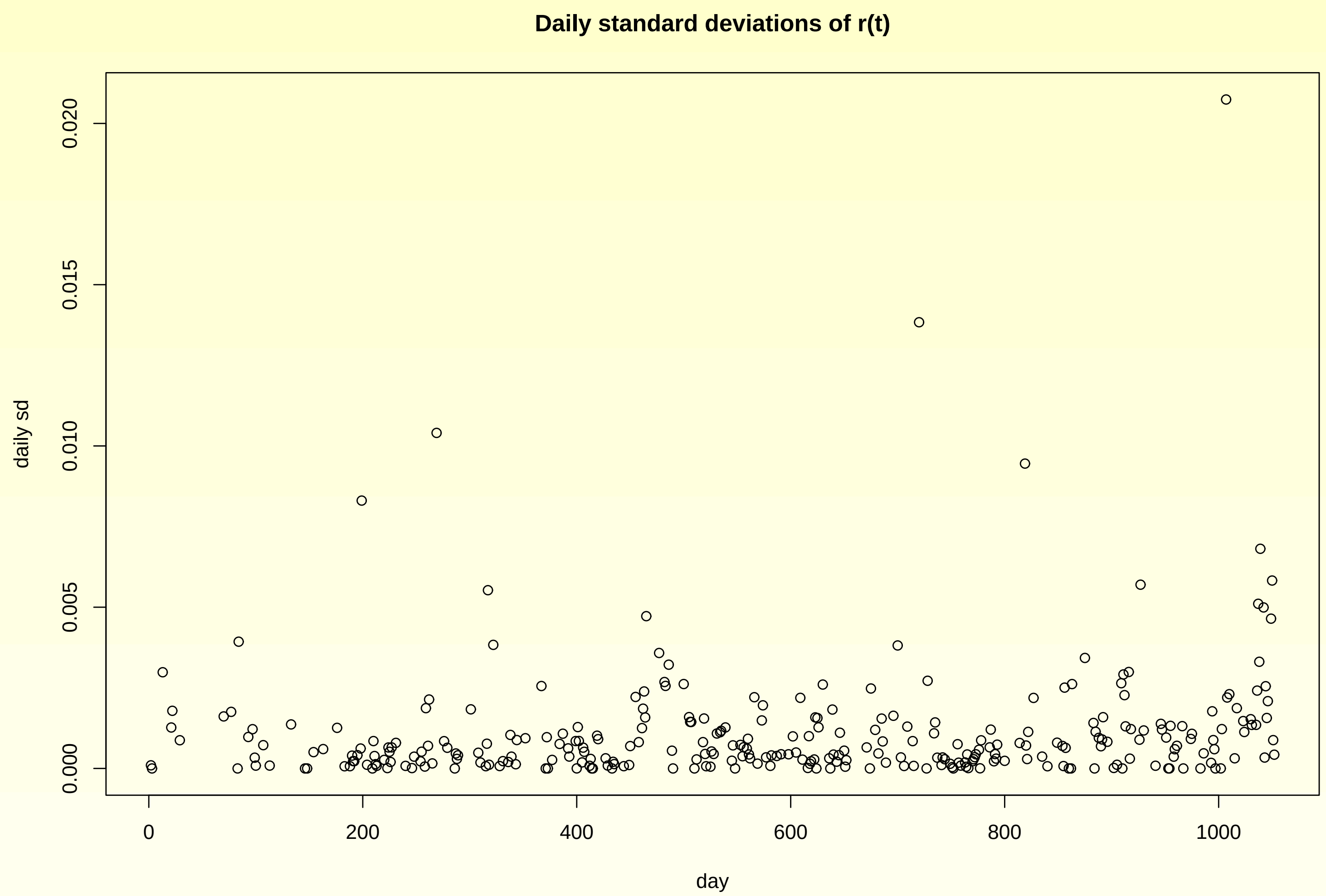

Figure 3: Daily standard deviations

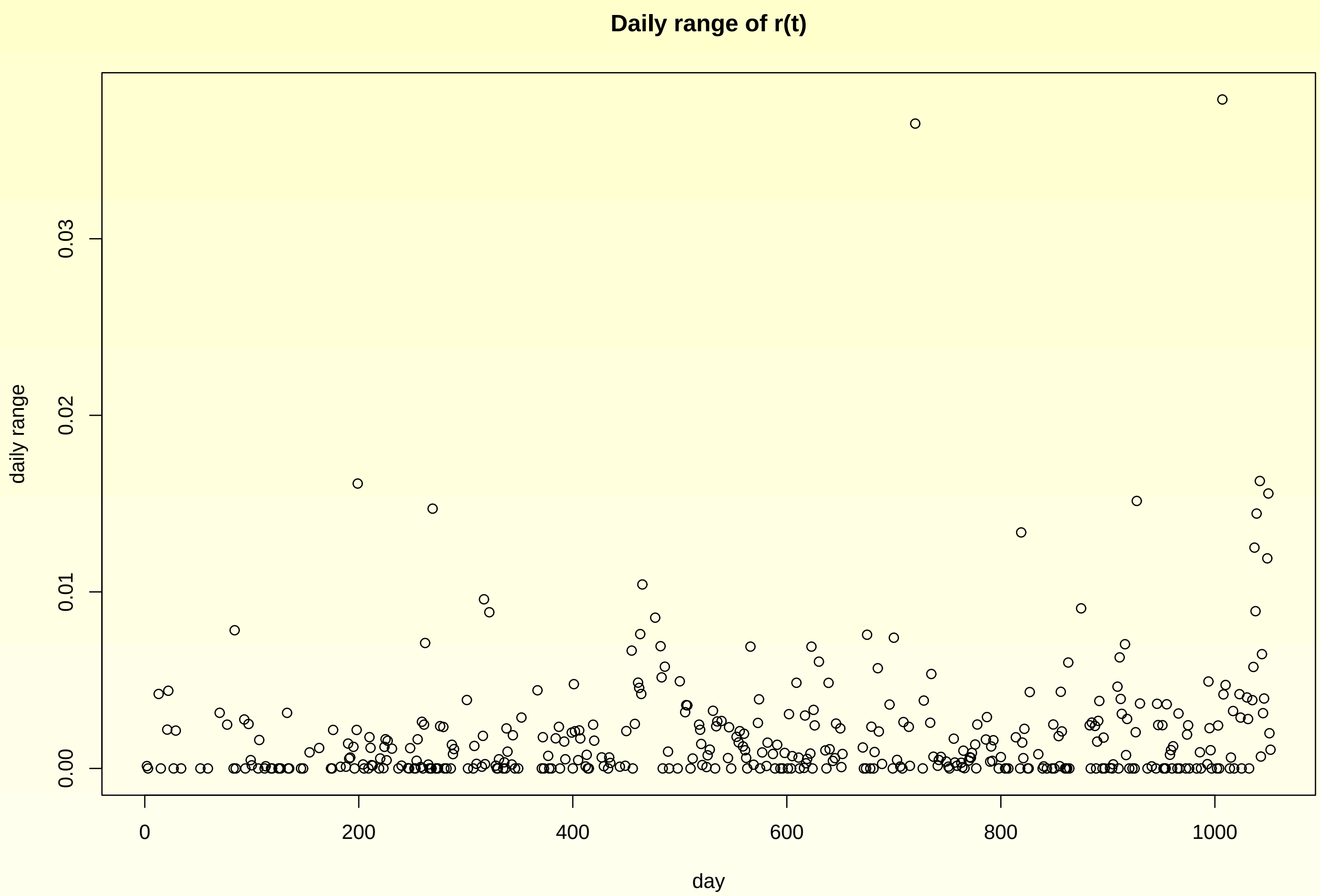

Figure 4: Daily observed range

- Of the 1933 observations, 664 had $\Delta_i = 0$. Many had very small $\Delta_i = 0$.

- The variance of the prices of the simultaneous observations offers a possibility to estimate the market microstructure noise. The standard deviation of simultaneous transactions is 6.4 points. (1%=100 points).

- For the expansion framework to work, the Δ_i's have to be small, but not to small, e.g., 10^{-8} is to small. The diffusion models rule out large jumps in small intervals.

2 Result of CIR for Icelandic data

- t=1 month. Median r(t) of a day chosen.

$\hat{\kappa}$ =15.090824	$\hat{\alpha}$ =0.046586	$\hat{\sigma}$ =0.081569
s.e.=0.028743	s.e.=0.000007	s.e.=0.000237

Looks much more peaceful than the test example that Chan, Karolyi, Longstaff & Sanders (1992) claim is a natural result.

Test example

T=1000, $\Delta = 0.04$, simulation result (1000 time-periods), one replication. Parameter values from Chan, Karolyi, Longstaff & Sanders (1992).

κ =0.24	$alpha$ =0.08	σ =0.08838
$\hat{\kappa}$ =0.23908	$\hat{\alpha}$ =0.08353	$\hat{\sigma}$ =0.08757
s.e.=0.02196	s.e.=0.00335	s.e.=0.00039

- $\bar{x} = 0.08358$, $\mu = \alpha = 0.08$
- $s = 0.03556$, $max(x(t)) = 0.2801$, $min(x(t)) = 0.0089$
- St.dev of stationary dist.=$\sqrt{\alpha * \sigma^2 / (2 * \kappa)} = 0.03608$

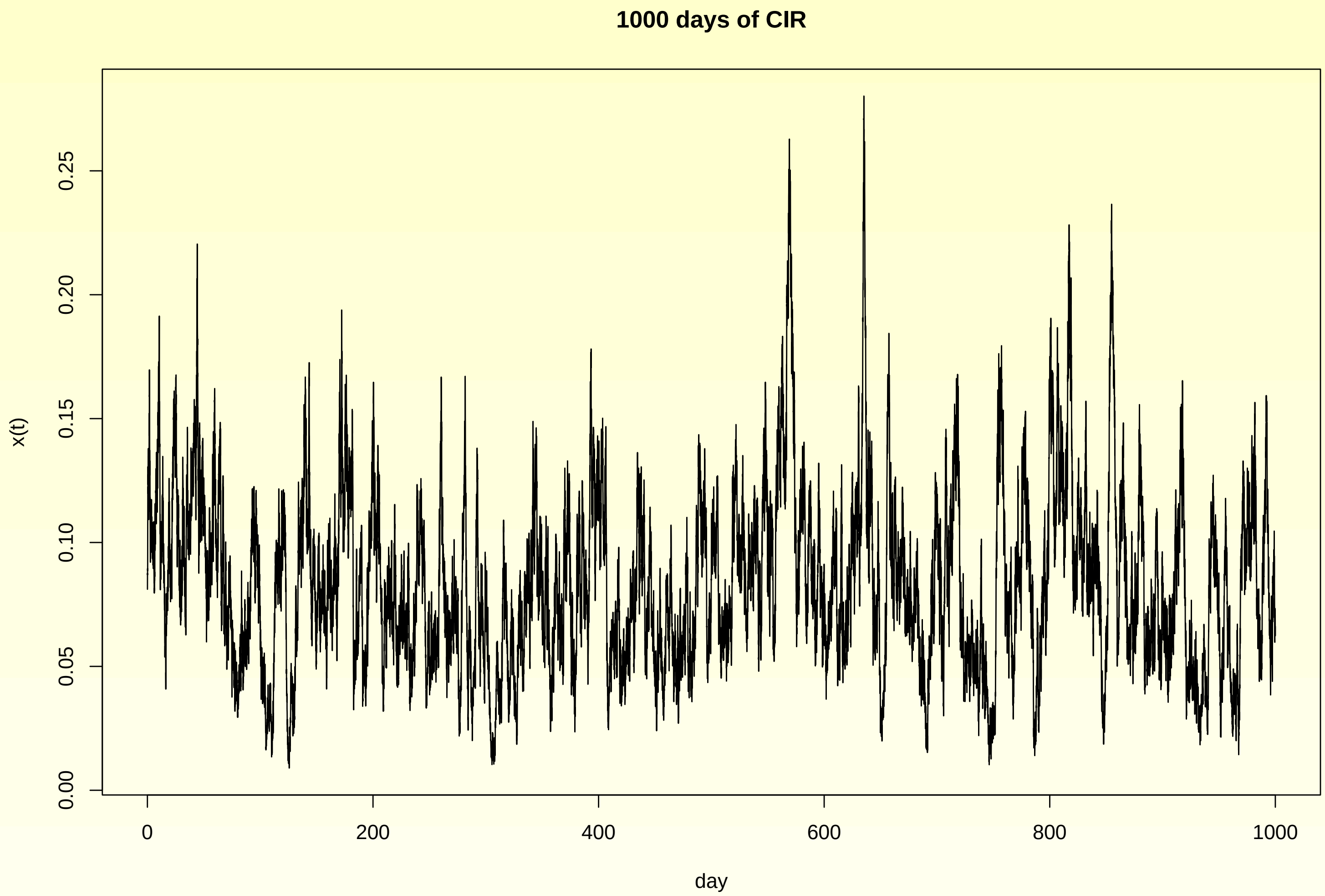

Figure 5: A simulated CIR for 1000 time-periods.

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References

Ait-Sahalia, Y. (1999). Transition densities for interest rate and other nonlinear diffusion. Journal of Finance, 54(4), 1361–1395.

Ait-Sahalia, Y. (2002). Maximum likelihood estimation of discretely sampled diffusions: A closed-form approximation approach. Econometrica, 70(1).

Chan, K., Karolyi, G., Longstaff, F., & Sanders, A. (1992). An empirical comparison of alternative models for the short-term interest rate. Journal of Finance, 47, 1209–1228.